J Glob Optim (2010) 48:3–15
DOI 10.1007/s10898-009-9523-y

Methods of embedding-cutting off in problems of mathematical programming

Valerian Bulatov

Received: 24 December 2009 / Accepted: 25 December 2009 / Published online: 7 January 2010

Abstract A class of functions, which reach their minima on a compact subset of the n-dimensional Euclidean space E^n, is considered. This is a rather wide class of functions, which are stable with respect to operations traditional for optimization problems. Functions of this class represent a tool suitable from the viewpoint of formal description of various applied problems. Furthermore, it is possible to develop rather efficient techniques for the purpose of finding global minima of such functions on a compact set. One of such techniques is discussed in Sect. 3 of the present paper.

Keywords Global optimization · Concave minorant · Nondegenerate matrix · Section plane

PACS 90C30

1 Introduction

The present paper discusses an approach to solving the problem, which is presented below, with the use of embedding-cutting (truncation) methods defined herein. Suppose that we have to obtain

$$\overline{x} = \arg\min\{\varphi(x) : x \in R\} \tag{1}$$

and

$$p = \varphi(\overline{x}),$$

where $\varphi(x)$ is a scalar continuous function of the vector $x \in R$; $R \subset E^n$ is a compact set. Minimization of $\varphi(x)$ on R is understood in the global sense.

V. Bulatov (✉)
Melentiev Energy Systems Institute SB RAS, Irkustk, Russia
e-mail: bulatov@isem.sei.irk.ru

It is obvious that methods of defining function $\varphi(x)$ and set R attribute problem (1) to some or another class of the theory of extremum problems.

It is also obvious that numerical methods used for solving problem (1) shall take account of the specificity of defining both function $\varphi(x)$, and set R. As a rule, efficiency of these methods is substantially dependent on the extent, to which this specificity is taken into account.

Let set R in problem (1) be defined explicitly and possess sufficiently good properties. Hence the principal difficulties of solving this problem are obviously related to either the "transcendent" properties of the function to be minimized (non-differentiability, discontinuity, the "ravine-type" character of the level lines, etc.) or to the technique of its defining.

Let us define the epigraph of function $\varphi(x)$ as a set

$$R_{n+1} = \{x, x_{n+1} : x \in R,\ \varphi(x) \leq x_{n+1}\}.$$

Obviously, problem (1) is equivalent to the following one: find

$$\overline{x} = \arg\min\{x_{n+1} : x, x_{n+1} \in R_{n+1}\}. \tag{2}$$

The function to be minimized in problem (2) is linear. Hence if the difficulty of solving problem (1) is bound up with the "bad" properties of $\varphi(x)$, then the difficulty of solving problem (2) is caused by the respective properties of the epigraph of function $\varphi(x)$. The iterative process related to solving problem (2) operates with some approximation of the epigraph $\varphi(x)$, rather than with this epigraph itself.

Let set $R^k_{n+1} \supset R_{n+1}$ and x^k, x^k_{n+1} be found. Define $R^{k+1}_{n+1} \supset R_{n+1}$ such that $x^k,\ x^k_{n+1} \notin R^{k+1}_{n+1}$ and

$$x^{k+1}_{n+1} = \min\{x_{n+1} : x,\ x_{n+1} \in R^{k+1}_{n+1}\}. \tag{3}$$

If there exists $K_1 \subset K = \{k : k = 1, 2, \ldots\}$ such that

$$\lim_{k\to\infty} x^k_{n+1} = p, \quad k \in K_1,$$

then the iterative process (3) may be called a method of sequential embedding the epigraph of the objective function.

Definition (3) of the method of sequential embedding the epigraph of the objective function is, generally speaking, not sufficiently pithy to be used as the only method for constructing algorithms and proving their convergence. Nevertheless, even relatively minor information on the methods related to defining the set R^j_{n+1} allows one to formulate the corresponding theorems of convergence of the iterative processes (3). One can find several confirming examples below.

Theorem 1 *Let sets R^k_{n+1} be defined as follows:*

$$R^k_{n+1} = \{x,\ x_{n+1} : x \in R,\ \varphi_k(x) \leq x_{n+1}\},$$
$$\forall k \in K = \{k : k = 1, 2, \ldots\},$$

where $\varphi_k(x)$ are continuous scalar functions of vector $x \in E^n$, such that

$$(1)\quad \psi_i(x^i) \leq \psi_k(x^k), \quad \forall k, \quad i \in K\,(k > i)$$

on set R;

$$(2)\quad \psi_k(x^i) \geq p, \quad \forall k, \quad i \in K\,(k > i) \tag{4}$$

where

$$x^i = \arg\min\{x_{n+1} : x,\ x_{n+1} \in R^i_{n+1}\},\quad i \in K.$$

Hence the iterative process (3) *represents (forms) a method of embedding the epigraph of the objective function.*

The proof may be found in [8].

Generally speaking, property (4) does not presume the inclusion of

$$R^1_{n+1} \supset \cdots\ R^i_{n+1} \supset \cdots\ R^k_{n+1} \supset \cdots \supset R_{n+1},\ (\forall k > i). \tag{5}$$

As a rule, inclusion (5) immediately results in (4). Consider an example of such an iterative process [11].

Theorem 2 *Let*

$$R^k_{n+1} = \{x, x_{n+1} : x \in R,\ \psi_k(x) \le x_{n+1}\},\quad \forall k \in K,$$

where

$$\psi_k(x) = \max_{1 \le i \le k} \Phi_i(x);$$

$\Phi_i(x)$ *are continuous scalar functions* $x \in E_n$, *such that*

$$\Phi_k(x^{k-1}) = \varphi(x^{k-1}),\quad \Phi_i(x) \le \varphi(x).$$

Hence the iterative process (3) *represents a method of sequential embedding the epigraph of the objective function.*

The Proof is rather trivial.

Below, methods of sequential embedding the feasible set are considered in parallel. Let us define a class of such methods.

Let vector x^k and set $R^k \supset R$ be found. Define $R^{k+1} \supset R$ such that $x^k \notin R^{k+1}$ and compute

$$x^{k+1} = \arg\min\{\varphi(x) :\ x \in R^{k+1}\}. \tag{6}$$

If there exists a set of indices $K_1 \subset K$ such that

$$\lim_{k \to \infty} x^k = \overline{x} \in R_r;\quad k \in K_1,$$

where R_r is a set of boundary points R, then the iterative process (6) may be called the method of sequential embedding of the feasible set in the sense of (6).

Introduction of the sequence of sets R^j similar to (3) is conditioned by the inconvenience of defining the initial set R, i.e. by the wish to conduct operations with the sets of some simpler nature. The structure and the methods of constructing the sequence $\{R^j\}$, are, as a rule, defined by the method of defining the initial set R. Moreover, the specific character of defining R is taken into account in the process of realization of sequence $\{R^j\}$. Furthermore, it determines the efficiency of the computational processes, which may be proposed, in many aspects.

Suppose set R is defined by the system of inequalities

$$R = \{x : g_j(x) \le 0,\quad j = \overline{1, m}\}.$$

Introduce the denotation $g(x) = \max_{1 \le j \le m} g_j(x)$. Hence problem (1) writes:

$$\min\{\varphi(x) : x \in R\},$$

where

$$R = \{x : g(x) \leq 0\}.$$

Assume that the minimum of $\varphi(x)$ is reached on the boundary of R. Let set R be embedded into some compact set R^0 having a simple structure. Suppose the minimum of $\varphi(x)$ is reached on the boundary of R^0. Let set $R^k \supset R$ and vector x^k be already found. Define set $R^{k+1} \supset R$ such that $x^k \notin R^{k+1}$ and compute

$$x^{k+1} = \arg\min\{\varphi(x) : x \in R^{k+1} \cap R^0\}. \tag{7}$$

If there exists a set of indices $K_1 \subset K$ such that

$$\lim_{k\to\infty} g(x^k) = 0, \quad k \in K_1, \tag{8}$$

then the iterative process (7) may be called the method of sequential embedding of the feasible set in the sense of (8). This technique of defining set R allows one to formulate theorems on convergence of the embedding methods without involving the property of convexity into consideration.

Theorem 3 *Let sets* R^k :

$$R^k = \{x : \psi_j(x) \leq 0, \ \ j = \overline{1, k}\},$$

where $\psi_j(x)$ *are continuous scalar functions such that* $\psi_{k+1}(x^k) = g(x^k)$. *The iterative process* (7), (8) *represents a method of sequential embedding of the feasible set in the sense of* (8).

Similar methods were investigated by different authors [10,15,16]. The methods allowing one to neglect inessential additional constraints were described in [13]. A systemic description of the embedding methods in convex programming can be found in [1]. The cutting (truncation) methods applied for the purpose of solving multi-extremum problems were probably described for the first time by Tuy [14]. The most general scheme similar to that of (3) can be found in [11]. Minimization methods for the concave function were considered, for example, in [5]. Some earlier results obtained by the author in this field were published in [2,1].

There exist other approaches to solving multi-extremum problems. These include (i) methods of reduction to one-parameter problems (the Peano-Strongin curve)[12], methods of covering and partition (decomposition) (Yevtushenko) [4], see also publications by Hansen [6], Jongen [7], etc.

2 Functions having a concave minorant on a compact set

Let there be given a compact set $R \subset E^n$ and a scalar function $f(x)$, which is that is defined everywhere on R, i.e. $f : R \to E^1$.

Definition 1 Let us speak that $f(x)$ has a concave minorant on R when there exists a function $\varphi(x, y)$, $\varphi : E^n \times E^n \to E^1$, which is continuous with respect to x for any fixed y and such that the following conditions hold:

1. $\varphi(x, y)$ is a concave function of x,
2. $f(x) \geq \varphi(x, y) \quad \forall x, y \subset R$,
3. $f(y) = \varphi(y, y) \quad \forall y \in R$.

The function $\varphi(x, y)$ shall be called the concave minorant of function $f(x)$ buil in at point y. Let us denote by $CM(R)$ the set of all functions $f(x)$, $f : A \to E^1$, $A \supset R$ having the concave minorant on set R, and the functions $f \subset CM(R)$ themselves shall be called the c.m.-functions.

It can easily be shown that linear, continuous convex and concave functions are c.m.-functions. In other words, the set of c.m.-functions includes linear functions, as well as continuous convex and concave functions. Let us describe some properties of c.m.-functions.

Theorem 1 *Let f and f_i $(i = 1, 2, \ldots, m)$ be c.m.-functions. Hence the following statements hold.*

(i) *Any nonnegative linear combination of functions f_i $(i = 1, 2, \ldots, m)$ is a c.m.-function;*
(ii) $\max\limits_{1\le i\le m} f_i(x)$ *and* $\min\limits_{1\le i\le m} f_i(x)$ *are c.m.-functions;*
(iii) $f^+(x) = \max\{0, f(x)\}$ *and* $f^-(x) = \min\{0, f(x)\}$ *are c.m.-functions.*

The proof of this statement can be found in [3].

The following result is the theoretical ground for our further investigations related to the problem of optimization.

Theorem 2 *Any function $f \in CM(R)$ is lower semicontinuous on R.*

The proof of this statement can be found in [3].

Consider the following problem of mathematical programming.

$$\min f(x), \tag{9}$$

$$g_i(x) \le 0, \quad i = 1, 2, \ldots, m, \tag{10}$$

$$x \in R, \tag{11}$$

where $R \subset E^n$ is a compact set and $f, g_i \in CM(R)$ $(i = 1, 2, \ldots, m)$. Due to Theorem 2, functions g_i $(i = 1, \ldots, m)$ are lower semicontinuous on R; and so, the sets $G^i = \{x \in R : g_i(x) \le 0\}$ are compact. Hence $T = G^1 \cap G^2 \cap \cdots \cap G^m \cap R$, which is the feasible set of problem (9)–(11), is also compact. Since function f is also lower semicontinuous, due to the well-known Weierstrass theorem, the lower semicontinuous function reaches its minimum on the compact set. Hence problem (9)–(11) has a finite solution.

From now on the problem (9)–(11) is called the c.m.-programming problem. Obviously, problems of linear, convex and concave programming are particular cases of the c.m.-programming problem.

3 The cutting method E^{n+1} for minimizing the c.m.-function on a polyhedron

We intend to describe the method intended for solving the following problem of mathematical programming:

$$\min f(x), \tag{12}$$

$$x \in R = \{x \in E^n : Ax \le b, \quad \alpha \le x \le \beta\}, \tag{13}$$

where $f \subset CM(R)$; A is an $m \times n$ matrix; $b \subset E^m$, $\alpha, \beta \subset E^n$. Denote by x^* one of solutions of problem (12), (13), and let $f^* = f(x^*)$.

We have to remind the reader of the fact that the following set

$$R_{n+1} = \{(x, x_{n+1}) : \ f(x) \le x_{n+1}, \quad x \in R\}$$

is called the supergraph of function $f(x)$ on R. Hence problem (12), (13) is equivalent to the following problem.

$$\min x_{n+1}, \tag{14}$$

$$(x, x_{n+1}) \subset R_{n+1}. \tag{15}$$

Consider now the method of solving problem (14), (15) proposed in [2].

Step 0. Suppose that the estimate $p \le f^*$ and the set $R^0_{n+1} \subset E^{n+1} : \ R^0_{n+1} \supset R_{n+1}$ are known (for example, we may take the set R^0_{n+1}, which has the form $R^0_{n+1} = \{(x, x_{n+1}) : \ x_i \ge \alpha_i, \ i = 1, 2, \dots, n, \ x_{n+1} \ge p\}$, where α and β are defined by (13)). Now assume $\varepsilon \ge 0$, put $f^0 = +\infty$ and $k = 1$, and define $R^1_{n+1} = R^0_{n+1} \cap \{x, x_{n+1} : x \in R\}$.

Step 1. Solve the following auxiliary problem:

$$\min x_{n+1}, \tag{16}$$

$$(x, x_{n+1}) \in R^k_{n+1}. \tag{17}$$

Let $y^k = (x^k, x^k_{n+1})$ be a solution of this problem.

Step 2. Determine $f^k = \min\{f^{k-1}, f(x^k)\}$ and $v^k : \ f(v^k) = f^k$. If

$$f^k - x^k_{n+1} \le \varepsilon, \tag{18}$$

the process of solving is terminated because ϵ^k is an optimal solution of problem (14), (15) (see Remark below). Otherwise, proceed to Step 3.

Step 3. Define the cone

$$\overline{R}^k_{n+1} = \{y \in E^{n+1} : \ \overline{A}^k y \le \overline{b}^k\},$$

which is formed by the constraints of problem (16), (17), which are active at point y^k, i.e.

$$\overline{A}^k y^k = \overline{b}^k. \tag{19}$$

When problem (16), (17) is nondegenerate, there exists a matrix $S^k = (\overline{A}^k)^{-1}$, which is inverse to matrix $\overline{A}^k$. The columns $S^{k,j} = (s^{k,j}, s^{k,j}_{n+1})$ of matrix S^k taken with the opposite sign are directional vectors of the cone's edges $\overline{R}^k_{n+1}$. Hence the real axes, which are described by the following equations

$$y^{k,j} = (x^{k,j}, x^{k,j}_{n+1}) = y^k - \lambda^{k,j} s^{k,j}, \ \lambda^{k,j} \ge 0, \quad j = 1, 2, \dots, n+1, \tag{20}$$

define the edges of cone $\overline{R}^k_{n+1}$.

Step 4. Let us construct a concave minorant $\varphi_k(x) = \varphi(x, x^k)$ of function $f(x)$ at point x^k. Let us find the numbers

$$\overline{\lambda}^{k,j} : \ \varphi_k(x^k - \overline{\lambda}^{k,j} s^{k,j}) = x^k_{n+1} - \overline{\lambda}^{k,j} s^{k,j}_{n+1}, \quad j = 1, 2, \dots, n+1. \tag{21}$$

Hence the points

$$z^{k,j} = (z^{k,j}, z^{k,j}_{n+1}) = y^k - \overline{\lambda}^{k,j} s^{k,j}, \quad j = 1, 2, \dots, n+1, \tag{22}$$

are the points at which the real axes (20) intersect with the surface $\varphi_k(x) = x_{n+1}$. Let us construct a plane

$$H^k y = h^k x + h^k_{n+1} x_{n+1} = g^k \tag{23}$$

passing through the points of (22).

Step 5. Define the set

$$R^{k+1}_{n+1} = R^k_{n+1} \cap \{y \in E^{n+1} : \ H^k y \le g^k\} \tag{24}$$

Put $k := k + 1$ and proceed to Step 1.

As shown in [1], the cuts determined by plane (23) are correct, i.e.

$$H^k y \le g^k \ \forall k, \quad \forall y \in R_{n+1}.$$

Remark Due to (24), we have $\forall k \ R^k_{n+1} \supset R^{k+1}_{n+1} \supset R_{n+1}$. Therefore, the solutions to the auxiliary problem (16), (17) satisfy the inequalities

$$x^1_{n+1} \le \cdots \le x^k_{n+1} \le x^{k+1}_{n+1} \le \cdots \le f^*. \tag{25}$$

On the other hand, due to the construction, we have $f^k \ge f^{k+1} \ge \cdots \ge f^*$. Hence the following bilateral estimate is valid:

$$x^k_{n+1} \le f^* \le f^k \quad \forall k. \tag{26}$$

Obviously, as soon as inequality (18) is satisfied, the algorithm terminates its operation at Step 2 because an ε-optimal solution has been obtained.

Lemma 1 *The equation of the cutting (truncating) plane* (23) *may be written in the form:*

$$(1/\lambda^k)\, \overline{A}^k\, y = (1/\lambda^k) b^k - 1, \tag{27}$$

where $(1/\lambda^k) = (1/\lambda^{k,1}, \ldots, 1/\lambda^{k,n+1})$, *the numbers* $\lambda^{k,j}$, $j = 1, \ldots, n+1$ *are defined by* (21) *and the* $(n+1) \times (n+1)-$ *matrix* $\overline{A}^k$ *and the vector* $\overline{b}^k$ *are defined by* (19).

Now, let us proceed to describing some properties of this algorithm and grounding the fact of its convergence.

Lemma 2 *If* $int\ R \ne 0$ *and* $||H^k|| \ge \delta > 0$, $(k = 1, 2, \ldots)$ *then the following inequalities hold:*

(a) $h^k_{n+1} \le 0$,
(b) $h^k_{n+1}| \ge \sigma > 0, \ \sigma - const.$,
(c) $\dfrac{||h^k||}{|h^k_{n+1}|} \le C, \ C - const.$

The proof of this statement can be found in [3].

Due to Lemma 2 proved above, we'll henceforth write down the equation of the cutting plane (23) in the following form

$$h^k x - h^k_{n+1} x_{n+1} = g^k. \tag{28}$$

Now denote by $d(x, R) = inf\ \{||x - y||, \ y \in R\}$ the distance from point x to set R.

Lemma 3 *Let Eq. 23 of the cutting plane be written in the form (25), i.e.* $H^k = (1/\lambda^k)\,\overline{A}^k$ *and* $g^k = (1/\lambda^k)\,\overline{b}^k - 1$. *Let us define the value of*

$$\omega^k = d(y^k, p^k), \tag{29}$$

where $p^k = \{y \in E^{n+1} : \ H^k y = g^k\}$. *Hence we have:*

$$\omega^k = ||H^k||^{-1}.$$

The proof of this statement can be found in [3].

The following result is well-known (see [9,1]). Furthermore, it represents an obvious corollary of Lemma 3 considered above.

Theorem 3 *If* $||H^k|| \leq L, \ L - const. \ \ \forall k$, *then the algorithm described above is finite.*

The condition $||H^k|| \leq L$ may be formulated in an equivalent form, which is more convenient from the viewpoint of further reasoning. Let us represent H in the following form $H^k = (h^k, -h^k_{n+1})$, where h^k and $-h^k_{n+1}$ are defined by (26).

Lemma 4 *Assume that*

$$0 < \gamma \leq ||h^k|| \leq \Gamma < +\infty, \quad \forall k \tag{30}$$

where γ *and* Γ *are constants. Hence* $||H^k|| \leq L, \ L - const.$ *if and only if* $\frac{||h^k||}{h^k_{n+1}} \geq \sigma > 0, \ \sigma - const.$

The proof of this statement can be found in [3].

Corollary 1 *If conditions (30) are satisfied then* $||H^k|| \to \infty$, *when* $k \to \infty$ *if and only if* $\frac{||h^k||}{h^k_{n+1}} \to 0, \ as \to \infty.$

Condition (30) is not strong rigorous. It is satisfied in practice almost always.

The condition $||H^k|| \leq L$, which ensures the finiteness of the algorithm, is rather important. However, it is often violated in practical computations. In the present case, the situation is as follows: the process turns out to be infinite. Hence $||H^k_{n+1}|| \to \infty$, when $k \to \infty$. Due to Lemma 3, this means that the cuts become inefficient (in the sense that $\omega^k \to 0$). So, due to Corollary 1, the cutting planes become "almost" parallel to each other and "almost" orthogonal to the vector $e^{n+1} \subset E^{n+1}$, where $e^{n+1} = (0, \ldots, 0, 1)$. However, the inequality $||H^k|| \leq L$ is not an obligatory condition from the viewpoint of the algorithm's efficiency. Therefore, the following theorem is valid.

Theorem 4 *Let the values of* ω^k, *defined by (29) be such that* $\lim_{k\to\infty} \omega^k = 0$ *and the series* $\sum_{k=1}^{\infty} \omega^k$ *diverges. Hence the algorithm considered above is finite.*

The proof of this statement can be found in [3].

Denote by $B(x, r)$ a ball of radius r, which has its center at point x. Let $\omega^0 \in R$. Hence, due to compactness of set R, there exists a sphere of smaller radius r_0, which contains R its center being located at point ω^0. Consider the function

$$\psi_k(x) = \frac{1}{h^k_{n+1}}(h^k x - g^k),$$

(see (26)) and define the quantity

$$\psi_k^r = \max\{\psi_k(x) : x \in B(\omega^0, r)\}.$$

The following theorem defines both necessary and sufficient conditions of convergence for the above algorithm.

Theorem 5 *Let $p \leq f^*$ and $\omega^0 \in R$. Hence $\lim_{k\to\infty} x_{n+1}^k = p$ if and only if there exists a finite $r \geq r^0$ and an index set $J \subset \{1, 2, \ldots.\}$ such that*

$$\lim_{\substack{j\to\infty \\ j\in J}} \psi_j^r = p.$$

The proof of this statement can be found in [3].

Corollary 2 *If the sufficient condition of Theorem 5 is satisfied and $p \geq f^* - \varepsilon$, where $\varepsilon > 0$, then the algorithm converges to an ε-optimal solution of the initial problem.*

Corollary 3 *Let $z^{k,j} = (z^{k,j}, z_{n+1}^{k,j})$, $j = 1, \ldots, n+1$ be the vectors defined by (22) and $\max_{1\leq j\leq n+1} z_{n+1}^{k,j} \geq f^*$. Hence $\lim_{k\to\infty} x_{n+1}^k = f^*$.*

In the general case, there may exist no finite r satisfying conditions of Theorem 5. However, Corollaries 2 and 3 allow one to construct reasonably efficient and convergent modifications of the cutting method considered above.

4 Some shift of the cutting plane

Our computations give evidence that sequence $\{x_{n+1}^k\}$ practically does not increase as the number of the iteration steps grows. The point is that the planes forming $\overline{R}_{n+1}^k$ become almost parallel to each other. To the end of improving the convergence of the method, the authors have proposed a number of modifications of the basic method intended for solving problem (12), (13). These modifications allow one to accelerate the convergence. Now, consider some of such modifications.

Let $|x_{n+1}^{k+1} - x_{n+1}^k| \leq \varepsilon > 0$. Let $R_{n+1}^k = \{y : \ E^{n+1} : \ \overline{A}^k y \leq \overline{b}^k\}$ represent the cone formed by the constraints of problem (16), (17), which are active at point y^k, and let $\overline{A}^k$ be a nonsingular $(n+1) \times (n+1)-$matrix.

Define the set $L_{n+1}^k = R_{n+1}^k \cap \{y : H^k y \leq g^k\}$. Now, having denoted by $\overline{A}^{kj}$ $(j = \overline{1, n+1})$ the j-th row of matrix $\overline{A}^k$, we can solve $(n+1)$ problems of linear programming (LP) of the following form:

$$\max\{(A^j)^T y : \ y \in L_{n+1}^k\}. \tag{31}$$

Suppose that $\tilde{y}^j$ be solutions of problem (31). Hence, obviously, the cone is Let $\tilde{y}^j$ be solutions to problems (31). Then, we have

$$\tilde{R}_{n+1}^k = \{y \in E^{n+1} : \ \overline{A}^{kj}(y - \tilde{y}^j) \leq 0, \ (j = \overline{1, n+1}\} \subset \overline{R}_{n+1}^k.$$

Now let us solve the following auxiliary problem of linear programming.

$$\min\{x_{n+1} : \ x, x_{n+1} \in \{R_{n+1}^k \backslash \overline{R}_{n+1}^k\} \cap \tilde{R}_{n+1}^k\}. \tag{32}$$

It is obvious that its solution satisfies the relation $\tilde{y}^k : \tilde{x}^k_{n+1} \geq x^k_{n+1}$. Let us construct the concave minirant $\tilde{\varphi}_k(x) = \varphi(x, \tilde{x}^k)$ at point $\tilde{y}$ and find the points at which the real axes $\tilde{y}^{kj} = \tilde{y}^k - \lambda^{kj} s^{kj}$, $\lambda^{kj} \geq 0$, (which are edges of the cone $\tilde{R}^k_{n+1}$) intersect with the surface $\tilde{\varphi}_k(x) = x_{n+1}$. The points $\tilde{y}^{kj}$ determine the appropriate cutting semispace $\tilde{H}^k\ y \leq \tilde{g}^k$ such that

$$\{H^k y \leq g^k\} \supset \{\tilde{H}^k\ y \leq \tilde{g}^k\}. \tag{33}$$

Having replaced the last inequality $H^k y \leq g^k$ in (24) with $\tilde{H}^k\ y \leq \tilde{g}^k$, we proceed to Step 1 of the main algorithm (see Sect. 3). But before proceeding to Step 1, one may try to improve the lower estimate $\tilde{x}^k_{n+1}$, obtained above. This may be done by repeating the corresponding procedures and by extending (constructing) the sequence $\tilde{x}^k_{n+1} \leq \tilde{x}^{k+1}_{n+1} \leq \ldots$ until this is expedient.

One may also try to improve the sequence of values of lower estimates by solving all the similar linear programming problems (31), which correspond not only to the constraints active at point y^k, rather than by solving only $(n+1)$ linear programming problems of the same type.

5 Second-order cuts

Here, we introduce a class of deeper cuts, which we qualify as the second-order cuts. We are still interested in solving problem (12), (13). Furthermore, we assume that $f(x)$ is a concave function.

Suppose, as before, that y^k solves problem (16), (17). Having presumed the nonsingular character of vertex y^k, /; $j = \overline{1, n+1}$, let us find the verteces y^{kj} $(j = \overline{1, n+1})$, which are neighboring to y^k. Suppose that $\overline{A}^{kj}$ are nonsingular $(n+1) \times (n+1)-$ matrices, which are active at points y^{kj}; of constraints, i.e. $\overline{A}^{kj}\ y^{kj} \leq \overline{b}^{kj}$.

Let us write down equations of the real axes passing from points y^{kj} to the polyhedron's vertices R^k_{n+1} adjacent to y^{kj} :

$$y^{kji} = y^{kj} - \lambda^{kji} S^{kji}, \quad j = \overline{1, n+1}, \quad i = \overline{1, n+1}, \quad \lambda^{kji} \geq 0. \tag{34}$$

These equations define edges of the cones $\overline{A}^{kj}\ y^{kj} \leq \overline{b}^{kj}$ related to vertices y^{kj} of polyhedron R^k_{n+1}.

Define $\overline{\lambda}^{kji} = \min\{\lambda^{kji}_1, \lambda^{kji}_2\}$, where λ^{kji}_1 corresponds to the point, at which the $i-$th edge intersects with the plane, which—in turn—corresponds to the vertex adjacent to y^{kj}, while λ^{kji}_2 corresponds to the point at which this edge intersects with the surface $f(x) = x_{n+1}$. As a result, we obtain the following matrix:

$$\begin{vmatrix} \overline{y}^{k1,1} & \overline{y}^{k1,2} & \ldots & \overline{y}^{k1,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{y}^{kn+1,1} & \overline{y}^{kn+1,2} & \ldots & \overline{y}^{kn+1,n+1} \end{vmatrix}, \tag{35}$$

whose elements are represented by vectors $\overline{y}^{kij}$. The first row in this matrix represents a set of vertices adjacent to y^{k1} or a set of points, at which the edges passing from y^{k1} intersect with the surface $f(x) = x_{n+1}$. The last row of the matrix (35) represents a set of vertices adjacent to $y^{k,n+1}$ or a set of the points at which the edges passing from $y^{k,n+1}$ intersect with the surface $\varphi(x) = x_{n+1}$. Each of the vertices y^{kj} has its adjacent vertex y^k. Suppose, this fact corresponds to the last column in (35).

Now, let us construct the planes

$$\tilde{H}^{kj} y = \tilde{h}^{kj}, \quad j = \overline{1, n+1} \tag{36}$$

passing through the points $\{\bar{y}^{kj,1}, \bar{y}^{kj,2}, \ldots, \bar{y}^{kj,n}, \bar{y}^k\}$. It is obvious that—likewise for (24)—the semispaces $\tilde{H}^{kj} y \le \tilde{h}^{kj}$ are appropriate, i.e. these contain R_{n+1}. Define the cone

$$\overline{H} = \{y : H^{kj} y \le \tilde{h}^{kj}, \quad \forall j = \overline{1, n+1}\} \tag{37}$$

and write down equations of the real axes drawn from points $x^{k,j}$ in the directions of its edges. Next, likewise for the system (21) and (22), let us find the points $z^{k,j}$ at which these real axes intersect with the surface $f(x) = x_{n+1}$.

Likewise for (23), let us draw the plane $H^{kj} = g^k$ through the points $z^{k,j}$. According to (24) define set R^{k+1}_{n+1}, put $k := k+1$, and proceed to Step 2 of the main algorithm (see Sect. 3). The geometric considerations give evidence that the last cut constructed is appropriate, correct.

The method proposed may be immediately extended to the cease of global minimization of functions. To this end, the surfaces $f(x) = x_{n+1}$ in the formulas of Sect. 5 shall be replaced with the surfaces $\varphi_k(x) = x_{n+1}$ given by concave minorants.

6 A numerical experiment

The cutting (truncation) method proposed in the present paper has been preliminary tested on a small set of problems of the form (12), (13). In the capacity of the ZZZ function we have considered the following nonconvex quadratic function

$$f(x) = x^T Qx + c^T x, \tag{38}$$

where Q is an indefinite symmetric matrix, and $c \in E^n$. The quadratic function (38) may be represented in the following form:

$$\begin{aligned} f(x) &= f^+(x) + f^-(x), \\ f^+(x) &= x^T Dx + c^T x, \quad f^-(x) = x^T Gx, \end{aligned} \tag{39}$$

where D is a positive semidefinite matrix and G is a negative definite matrix. It can readily be seen that, in this case, $f^+(x)$ is a convex function, while $f^-(x)$ is a concave one. Let y, be a feasible point; the concave minorant $\varphi(x, y)$ be determined as follows:

$$\varphi(x, y) = f^+(y) + \nabla f^+(y)^T (x - y) + f^-(x) = 2y^T Dx - y^T Dy + c^T x + x^T Gx.$$

The test examples were generated randomly. The complexity of a particular test example was determined by the number of negative eigenvalues of matrix Q and by their values. Noteworthy, the values of negative eigen-numbers, which define the curvature of the concave part of the objective function, played a substantially smaller role than their number.

Our cutting (truncation) algorithm was organized as follows. Firstly, cuts in E^{n+1} were conducted. If the difference between the sequential optimum values of auxiliary problems (16), (17) was smaller than 0.01, then the cutting plane was shifted according to the technique described in Sect. 4. After that, cuts were again conducted in E^{n+1}. By the time moment of

Table 1 Testing results of the cutting-method. For details see the text

n	k	N_1	N_s	N_2	L
2	2	56	0	0	3
3	2	98	0	0	3
3	3	156	1	0	3
4	2	254	1	1	3
4	3	312	2	2	3
4	4	437	3	2	3
5	3	496	3	3	3
5	4	663	3	3	3
5	5	935	3	3	3
7	5	1,344	3	3	2
7	7	2,101	3	3	2
8	6	4,349	3	3	2
8	8	6,562	3	3	1
9	7	9,874	3	3	1
9	8	12,104	3	3	1
10	8	18,465	3	3	1

applying the procedure of shifting the cutting plane, the current cone containing the supergraph of the objective function was formed by the "almost" parallel planes, and this shift did not change the structure of the cone but rather allowedone to move the operation of the cutting algorithm on, while jumping over a definite number of iteration steps. Repeated applications of the procedure of shift did not give any practical effect. After that, the second-order cut described in Sect. 5 was constructed. The objective of such second-order cuts is to make the current cone "more pointed" and hence make the cuts performed in E^{n+1} deeper. The efficiency of the second-order cuts is higher than that of the shift procedure. At least, the cuts may be applied several times (as a rule, two or three times). After the second-order cuts, cuts in E^{n+1} were performed again.

Therefore, the algorithm to be tested included the following stages. First of all, cuts in E^{n+1} were conducted. Next, a shift of the cutting plane followed. After that, cuts in E^{n+1} were conducted again. Next, second-order cuts were constructed. After that, the entire process was repeated from the very beginning. The operation of the algorithm was terminated in case when none of the approaches gave any practical progress, i.e. irrespectively of the procedures applied, the sequential values of the auxiliary problems (16), (17) differed by less than 0.01.

If in course of operation of the algorithm three digits in the lower estimate coincided with three digits in the upper estimate of the bilateral estimate (26), then we presumed that an optimal solution was obtained.

The computations were conducted on a personal 1.5 GHz/400 MHz/512 MB Celeron computer. Some results of the tests are given in Table 1. The following denotations are used: n is the number of variables, k is the number of negative eigenvalues of matrix Q, N_1 is the total number of cuts, N_s is the number of shifts of the cutting plane, N_2 is the number of second-order cuts, and L is the number of identical digits in the upper and lower estimates in (26). The algorithm's execution time was rather small: even in case of the most difficult problems, it took less than 1 min for the algorithm to complete its operation. From the author's viewpoint, the execution time is of minor importance in our case. At the expense of optimization of the algorithm's internal procedures it is possible to reduce the execution time by two or even three times.

The results of testing have give evidence that the algorithm proposed represents an efficient tool for solving small-dimensional problems. The advantages of our approach are represented by its simplicity (in contrast to the branch and bound method) and by a better lower bound estimate of the global minimum than that proposed by other similar approaches.

Acknowledgment The work is supported by the grant of RFBR 09-01-00307.

References

1. Bulatov, V.P.: Embedding Methods in Optimization Problems. Nauka, Novosibirsk (in Russian) (1977)
2. Bulatov, V.P.: Methods for solving multiextremal problems (global search) (in Russian). In: Methods of Numerical Analysis and Optimization, pp. 133–157. Nauka, Novosibirsk (1987)
3. Bulatov, V.P., Khamisov, O.V.: Cutting methods in E^{n+1} for global optimization of a class of functions. J. Comput. Math. Math. Phys. **47**(11), 1756–1767 (2007)
4. Evtushenko, Yu.G.: Methods of the decision of extreme problems and their application in systems of optimization. M, p. 432. Nauka (1982)
5. Forgo, F.: Cutting plane methods for solving nonconvex programming problems. Acta. Cybern. **I**(3), 171–192 (1972)
6. Guddat, J., Jongen, H.Th., Ruckmann, J.-J.: On stability and stationary points in nonlinear optimization. J. Aust. Math. Soc. Ser. B **28**, 35–56 (1986)
7. Hansen, P., Jaumard, B., et al.: Global optimization one-dimensional Lipshicev's functions. In: Optimization: Models, Methods, Decisions, pp. 287–338 (in Russian). Nauka, Novosibirsk (1992)
8. Hirriart-Urruty, J.B.: From convex optimization to nonconvex optimization, Part 1: necessary and sufficient conditions for global optimality. In: Nonsmooth Optimization and Related Topics, pp. 219–239. Plenum, New York (1989)
9. Horst, R., Tuy, H.: Global Optimization. Deterministic Approaches. Springer, Berlin (1993)
10. Kelley, J.E.: The cutting-plane method for solving convex programs. SIAM J. **8**(4), 703–712 (1960)
11. Piyavsky, S.A.: An algorithm for searching for the absolute extreme of functions. ZhVM MF **12**(4), 888–896 (1972)
12. Strongin, R.G.: Numerical methods in multiextreme problems. M, p. 239. Nauka (1978)
13. Topkis, D.M.: Cutting plane methods without nested constraint sets. Operat. Res. **3**, 1437–1440 (1970)
14. Tuy, H.: Concave programming in linear constrains. Dokl. AN SSR **159**(1), 404–413 (1964)
15. Veinott, A.F.: The supporting hyperplane method for unimodal programming. Oper. Res. **16**(1), 147–152 (1967)
16. Zoutendijk, G.: Nonlinear programming: a numerical survey. SIAM J. Control **4**(1), 194–210 (1966)